Sums of Squares: An Elementary Method

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1 Introduction

If $x_1, x_2, \ldots, x_s$ are integers positive negative or zero such that

$$x_1^2 + x_2^2 + \cdots + x_s^2 = n,$$

then $(x_1, x_2, \ldots, x_s)$ is called a representation of n as a sum of s squares, and the total number of representations is denoted by $R_s(n)$. Two representations $(x_1, x_2, \ldots, x_s)$ and $(y_1, y_2, \ldots, y_s)$ are considered to be different unless

$$x_1 = y_1, x_2 = y_2, \ldots, x_s = y_s.$$

Further, using a notation introduced by J.W.L. Glaisher, we write $R_{\alpha,\beta}(n)$ for the number of representations of n as a sum of squares of which α are odd and β are even, no restriction being placed upon the order of the squares. Observe that $R_s(0) = 1$, and that $R_{\alpha,\beta}(0) = 1$, if $\alpha = 0$, but that otherwise $R_{\alpha,\beta}(0) = 0$.

Mathematicians have been interested in evaluating $R_s(n)$ and $R_{\alpha,\beta}(n)$ since the 17th century and earlier, as a study of Chapters 6–9 of [3] will confirm and various methods have been used. Nowadays the use of modular forms provides the simplest answer. In the case of $R_s(n)$ the problem boils down to expressing the s-th power of the theta function ϑ_3 as an Eisenstein series plus a cusp form, both being holomorphic modular forms of weight $s/2$. The problem is simpler when s is even, as we shall generally assume, mainly because no complicated multiplier systems are then involved. See, for example, §7.4 of [10].

In a series of papers in the old Quarterly Journal (summarised in [4]) Glaisher obtained formulae for $R_{2s}(n)$ and $R_{\alpha,\beta}(n)$ for $s = 1, 2, \ldots, 9$ and $\alpha + \beta = 2s$ by means of elliptic function equations. His method, now regarded as old-fashioned, was essentially based on the consideration of various power series involving ϑ_3 and the Jacobi functions k and k'. Among these there appeared a number of what we now call cusp forms, whose Fourier coefficients had interesting multiplicative, or partly multiplicative, properties; see [8]. Where these cusp forms were absent, or had a zero Fourier coefficient, $R_{2s}(n)$ or $R_{\alpha,\beta}(n)$ was expressed in terms of divisor functions of different types. Among Glaisher's formulae the only ones which do not involve arithmetical functions other than divisor functions are formulae (1)–(24) given below. The divisor functions are defined in equations (25) and (26). Then we have

$$R_2(n) = 4E_0(n), \tag{1}$$

$$R_4(n) = 8\{2 + (-1)^n\}\Delta_1(n), \tag{2}$$

$$R_6(n) = 16E_2'(n) - 4E_2(n), \tag{3}$$

$$R_8(n) = \begin{cases} 16\Delta_3(n) & (n \text{ odd}) \\ \frac{16}{7}\{8\Delta_3'(n) - 15\Delta_3(n)\} & (n \text{ even}) \end{cases} \tag{4}$$

If $N = 2^\alpha N_1$, where $\alpha \geq 0$ and $N_1 \equiv 3 \pmod 4$,

$$R_{10}(N) = \frac{4}{5}\{E_4(N) - 16E_4'(N)\}, \tag{5}$$

$$R_{12}(n) = \frac{24}{31}\{21\Delta_5(2n) + 10\Delta_5'(2n)\}, \tag{6}$$

$$R_{2,0}(8n+2) = 4E_0(4n+1), \tag{7}$$

$$R_{1,1}(4n+1) = 4E_0(4n+1), \tag{8}$$

$$R_{4,0}(8n+4) = 16\Delta_1(2n+1), \tag{9}$$

$$R_{3,1}(4n+3) = 8\Delta_1(4n+3), \tag{10}$$

$$R_{2,2}(4n+2) = 24\Delta_1(2n+1), \tag{11}$$

$$R_{1,3}(4n+1) = 8\Delta_1(4n+1), \tag{12}$$

$$R_{6,0}(8n+6) = -8E_2(4n+3), \tag{13}$$

$$R_{4,2}(4n) = 240E_2'(n), \tag{14}$$

$$R_{3,3}(4n+3) = -20E_2(4n+3), \tag{15}$$

$$R_{2,4}(4n+2) = 60E_2'(2n+1), \tag{16}$$

$$R_{8,0}(8n) = 256\Delta_3'(n), \tag{17}$$

$$R_{4,4}(4n) = 1120\Delta_3'(n). \tag{18}$$

If N and N_1 are as before equation (5), then

$$R_{8,2}(4N) = 576E_4'(N), \tag{19}$$

$$R_{6,4}(2N_1) = 168E_4'(N), \tag{20}$$

$$R_{4,6}(4N) = 2688E_4'(N), \tag{21}$$

$$R_{2,8}(2N_1) = 36E_4'(N_1), \tag{22}$$

$$R_{8,4}(8n) = 3960\Delta_5'(2n), \tag{23}$$

$$R_{4,8}(8n) = 3960\Delta_5'(2n). \tag{24}$$

In formulae (7)–(18), $R_{\alpha,\beta}(m) = 0$ unless m is of the form stated. This is obvious since, if x is odd, then $x^2 \equiv 1 \pmod 8$, while, otherwise, we have $x^2 \equiv 0 \pmod 4$. The functions $\Delta_\nu(n)$, $\Delta_\nu'(n)$, $E_\nu(n)$ and $E_\nu'(n)$ are defined as follows. Let δ be an odd divisor of n and write $\delta\delta' = n$. Also let $\chi(n)$ be the character defined by

$$\chi(n) = 0 \ (n \text{ even}), \quad \chi(n) = (-1)^{(n-1)/2} (n \text{ odd}).$$

Then

$$\Delta_\nu(n) = \sum_{\delta|n} \delta^\nu, \quad \Delta'_\nu(n) = \sum_{\delta|n} (\delta')^\nu, \tag{25}$$

$$E_\nu(n) = \sum_{\delta|n} \chi(\delta)\delta^\nu, \quad E'_\nu(n) = \sum_{\delta|n} \chi(\delta)(\delta')^\nu, \tag{26}$$

Note that formulae for $R_{0,2s}(4n)$ are omitted since $R_{0,2s}(4n) = R_{2s}(n)$.

2 Historical Remarks

Some of these formulae have, over the years, been proved by elementary methods not involving elliptic or modular functions, by mathematicians such as Pepin, Dirichlet and Uspensky. The most complete elementary account so far published is Helmut Bessel's doctoral dissertation submitted to the University of Königsberg [2] in 1929. He bases his work on three formulae of Liouville in articles in the Journal de Mathématiques between the years 1840 and 1850, with the title 'Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres'. See Chapter 11 of [3] or pp. 365–463 of [1]. I give the following as an example:

$$\sum\{f(x+y) - f(x-y)\} = \frac{1}{2}\sum_{\delta|n} \delta'\{f(0) - f(\delta')\},$$

where f is an odd function and the summation is over all solutions of the equation $ax+by = 2n(a, b, x, y \text{ odd})$.

By means of these formulae Bessel proves the majority of the results stated above, excluding formulae (19)–(24). He also makes use of certain cusp form coefficients such as $\sum(x_1^4 - 3x_1^2x_2^2)$, the summation, in this case, being taken over all solutions of the equation $x_1^2 + x_2^2 = n$. This is a multiple of Glaisher's multiplicative function $\chi_4(n)$, which is the Fourier coefficient of a cusp form of weight 5, and arises in the study of 10 squares.

I completed the work of the present article in July 1944 but it was not until October 1947 that I managed to borrow a copy of Bessel's thesis, through Inter Library Loan. I have now, many years later, succeeded in obtaining a photocopy of his thesis through the courtesy of the Mathematics Library of the University of Illinois at Urbana.

In what follows I assume (1) and deduce the remaining formulae from it. A proof of (1) can be found in various treatises on number theory, for example in Chapter 2 of [5] or §6.7 of [7]. An interesting account of the history of the subject will be found in Chapter 9 of [6] and the notes to that chapter. Finally, a good account of Liouville's methods is in Chapter 13 of [11] and in [1].

In my treatment of the subject I do not use any of Liouville's formulae, but base my account on the elementary Theorem 1 below. It may be mentioned that this work has already been used in my paper [9]. Inevitably there are certain similarities between my method and Bessel's, but the two methods are not identical.

3 General Results

All letters, with the exception of π, denote integers; in particular $x, y, a, b, c, \xi, \eta, n, d$ and δ denote non-negative integers, and will usually be positive. We write d for any divisor of n and δ for an odd divisor of n, and put

$$d' = \frac{n}{d}, \quad \delta' = \frac{n}{\delta}. \tag{27}$$

It follows from (25) and (26) that, if $\alpha \geq 0$,

$$\Delta_\nu(2^\alpha n) = \Delta_\nu(n), \quad \Delta'_\nu(2^\alpha n) = 2^{\alpha\nu}\Delta'_\nu(n), \tag{28}$$

$$E_\nu(2^\alpha n) = E_\nu(n), \quad E'_\nu(2^\alpha n) = 2^{\alpha\nu}E'_\nu(n). \tag{29}$$

If n is odd,

$$\Delta'_\nu(n) = \Delta_\nu(n), \quad E'_\nu(n) = \chi(n)E_\nu(n). \tag{30}$$

In particular, by (26) and (30),

$$E_0(4n+3) = E'_0(4n+3) = 0. \tag{31}$$

The number of representations of a number as a sum of s squares may be expressed in terms of the representations by less than s squares by means of the formulae

$$R_{a+b}(n) = \sum_{m=0}^{n} R_a(m)R_b(n-m), \tag{32}$$

and

$$R_{a+b,c+d}(n) = \frac{(a+b+c+d)!a!b!c!d!}{(a+b)!(c+d)!(a+c)!(b+d)!}\sum_{m=0}^{n} R_{a,c}(m)R_{b,d}(n-m). \tag{33}$$

The proof of (32) is obvious. If $R'_{\mu,\nu}(n)$ is the number of representations of m in the form

$$x_1^2 + x_2^2 + \cdots + x_\mu^2 + y_1^2 + y_2^2 + \cdots + y_\nu^2,$$

where $x_1, x_2, \ldots, x_\mu$ are odd, and $y_1, y_2, \ldots, y_\nu$ are even, then clearly,

$$R_{\mu,\nu}(m) = \frac{(\mu+\nu)!}{\mu!\nu!}R'_{\mu,\nu}(m),$$

and (33) follows since

$$R'_{a+b,c+d}(n) = \sum_{m=0}^{n} R'_{a,c}(m)R'_{b,d}(n-m).$$

As an illustration of the use of formulae (32) and (33), we have

$$R_{8,0}(8n) = \sum_{m=0}^{n} R_{4,0}(m)R_{4,0}(n-m) = \sum_{m=1}^{n} R_{4,0}(8m-4)R_{4,0}(8n-8m+4).$$

If we assume the truth of (9) for the moment, we obtain

$$R_{8,0}(8n) = 256 \sum_{m=1}^{n} \Delta_1(2m-1)\Delta_1(2n-2m+1).$$

The right hand side may be written as $256 \sum xy$, where the summation is extended over all solutions a, b, x, y of the equation

$$ax + by = 2n$$

for which $abxy$ is odd. Thus the problem of evaluating $R_{8,0}(8n)$ is reduced to the investigation of the solutions of this equation. This example is typical of the methods used to derive (2)–(24). We denote by $T(n)$ the set of all positive a, b, x, y which satisfy

$$ax + by = n, \quad x, y \text{ odd},$$

and write $T_1(n)$ for the subset of $T(n)$ for which ab is odd. Clearly $T_1(n)$ is empty unless n is even.

Further, we denote by $S(n)$ the set of all positive ξ, η, x, y that satisfy

$$\xi x + \eta y = n, \quad xy \text{ odd}, \quad (\xi, \eta) = 1.$$

We write $T(n)$, $T_1(n)$, or $S(n)$ below the summation sign to indicate a sum carried out over all solutions of $T(n)$, $T_1(n)$, or $S(n)$, respectively. If there are no solutions (e.g. if $n = 1$), such a sum is empty and has the value zero.

Let $f(x, y, a, b)$ be any function of x, y, a, b. Put $d = (a, b)$, $a = d\xi$, $b = d\eta$. Then we have

$$\sum_{T(n)} f(x, y, a, b) = \sum_{\delta | n} \sum_{S(d')} f(x, y, d\xi, d\eta). \tag{34}$$

Similarly, if n is even,

$$\sum_{T_1(n)} f(x, y, a, b) = \sum_{\delta | n} \sum_{S(d')} f(x, y, \delta\xi, \delta\eta). \tag{35}$$

If

$$f(x, y, ca, cb) = c^{\nu} f(x, y, a, b) \tag{36}$$

for every c, (34) and (35) may be written

$$\sum_{T(n)} f(x, y, a, b) = \sum_{d|n} d^{\nu} \sum_{S(d')} f(x, y, \xi, \eta), \tag{37}$$

$$\sum_{T_1(n)} f(x, y, a, b) = \sum_{\delta|n} \delta^{\nu} \sum_{S(\delta')} f(x, y, \xi, \eta). \tag{38}$$

Equation (38) holds only if n is even.

The preceding formulae show that we need only consider solutions of $S(n)$.

Write

$$\delta_n = \frac{1}{2}\{1 - (-1)^n\}.$$

We assume that $n > 1$, as otherwise $S(n)$ is empty.

Let $C(n)$ be the set of solutions of $S(n)$ for which

$$|\xi - \eta| = \delta_n. \tag{39}$$

When n is odd we write $C(n) = C_1(n) + C_2(n)$, where $C_1(n)$ is the set of solutions of $C(n)$ for which $\xi = \eta - 1$, and $C_2(n)$ is the set for which $\xi = \eta + 1$.

The solutions of $C(n)$ are easy to obtain, and are given in the following three Lemmas.

Lemma 1 *If n is even $C(n)$ consists of the $\frac{1}{2}n$ solutions*

$$x = 2u - 1, \quad y = n - 2u + 1, \quad \xi = \eta = 1, \quad 1 \le u \le \frac{1}{2}n.$$

This follows, since $\xi = \eta$ by (39), and therefore $\xi = \eta = 1$, since $(\xi, \eta) = 1$.

Lemma 2 *If n is odd $C_1(n)$ consists of the solutions*

$$\xi = \left[\frac{n}{2t}\right], \quad \eta = \xi + 1, \quad x = 2t - y, \quad y = n - 2\xi t,$$

where t is any integer satisfying

$$0 < t < \frac{1}{2}n. \tag{40}$$

Write $2t = x + y$ for any solution of $C(n)$. Then clearly (40) holds, and the lemma follows since

$$2\xi t < 2\xi t + y = n < 2\xi t + 2t.$$

We have, similarly,

Lemma 3 *If n is odd $C_2(n)$ consists of the $\frac{1}{2}(n-1)$ solutions*

$$\xi = \eta + 1, \quad \eta = \left[\frac{n}{2t}\right], \quad x = n - 2\eta t, \quad y = 2t - x,$$

where t is any integer satisfying (40).

We write $c(n)$ for the number of members of $C(n)$. By Lemmas 1, 2, and 3,

$$c(n) = \begin{cases} \frac{1}{2}n & (n \text{ even}), \\ n - 1 & (n \text{ odd}). \end{cases} \tag{41}$$

We now consider the properties of another subclass of solutions of $S(n)$. Let $C'(n)$ be the set of solutions of $S(n)$ for which $x = y$. Then we have $x(\xi + \eta) = n$. Thus x must be a divisor of n. Let δ be any odd divisor of n. Then, if there is a solution of $S(n)$ with $x = y = \delta$, we must have, by (27),

$$\xi + \eta = \delta'. \tag{42}$$

It follows that $\delta' > 1$, i.e.

$$\delta < n. \tag{43}$$

If (43) is satisfied, there are exactly $\phi(\delta')$ solutions of (42); here ϕ is Euler's function. For, since $(\xi, \eta) = 1$, the only values which ξ can take are the $\phi(\delta')$ numbers less than δ' that are prime to δ'.

Lemma 4 *For any positive integer n*

$$\sum_{\delta|n} \phi(\delta') = \begin{cases} n & (n \text{ odd}), \\ \frac{1}{2}n & (n \text{ even}). \end{cases}$$

The case when n is odd follows from the well known result, which holds for odd and even n:

$$\sum_{\delta|n} \phi(d) = n.$$

Suppose that n is even of the form $2^\alpha m$, where m is odd and $\alpha > 0$. Then

$$\sum_{\delta|n} \phi(\delta') = \sum_{\delta|n} \phi(2^\alpha)\phi(m/d) = 2^{\alpha-1} \sum_{d|m} \phi(d) = 2^{\alpha-1} m = \frac{1}{2} n.$$

Lemma 5 *The solutions of $C'(n)$ are given by*

$$x = y = \delta, \quad \xi = u, \quad \eta = \delta' - u, \quad 0 < u < \delta',$$

where δ is any odd divisor of n which is less than n, $\delta' = n/\delta$, and $(u, \delta') = 1$. The total number of solutions is $c(n)$.

The first part of the lemma has already been proved, so that it remains to prove the second part. The number of solutions of $C'(n)$ is

$$c'(n) = \sum_{\delta|n, \delta<n} \phi(\delta').$$

By Lemma 4, if n is odd,

$$c'(n) = \sum_{\delta|n} \phi(\delta') - 1 = n - 1,$$

and, if n is even,

$$c'(n) = \sum_{\delta|n} \phi(\delta') = \frac{1}{2} n.$$

It follows from (41) that $c'(n) = c(n)$.

We now show that the solutions of $S(n)$ may be divided into sets, or 'chains', each associated with a unique solution (x, y, ξ, η) of $S(n)$.

Theorem 1 *The solutions of $S(n)$ may be divided into $c(n)$ chains of solutions such that each solution belongs to one and only one chain. Each chain consists of a sequence of solutions $(x_m, y_m, \xi_m, \eta_m)(m = 1, 2, \ldots, q)$, with the following properties:*

$$|\xi_1 - \eta_1| = \delta_n, \quad x_q = y_q. \tag{44}$$

Also, if $q > 1$, $1 \leq m < q$,

$$0 \leq |x_{m+1} - y_{m+1}| < |x_m - y_m|, \tag{45}$$

$$\delta_n \leq |\xi_m - \eta_m| < |\xi_{m+1} - \eta_{m+1}|, \tag{46}$$

$$x_{m+1} + y_{m+1} = |x_m - y_m|, \tag{47}$$

$$\xi_{m+1} - \eta_{m+1} = -(\xi_m + \eta_m)\operatorname{sgn}(x_m - y_m), \tag{48}$$

$$\chi(x_{m+1}) - \chi(y_{m+1}) = -\{\chi(x_m) + \chi(y_m)\}\operatorname{sgn}(x_m - y_m). \tag{49}$$

Proof: Since $n > 1$ there exists at least one solution for which

$$|\xi - \eta| = \delta_n, \quad x = y. \tag{50}$$

Any such solution we set in a chain by itself. Clearly (44) is satisfied. If $n = 2$ or 3, there are no solutions other than those given by (50), so that the theorem holds. □

We suppose, therefore, that $n > 3$. Then there exist solutions for which $|\xi - \eta| > 1$, and also solutions for which $x \neq y$. We consider those two cases separately in what follows.

Suppose that we have a solution with $x \neq y$. Write

$$\epsilon = \operatorname{sgn}(x - y).$$

We can generate a new solution by means of the following transformation,

$$\begin{aligned} x' &= \lambda x - (\lambda + \epsilon)y, \xi' = \lambda\xi + (\lambda - \epsilon)\eta, \\ y' &= -(\lambda - \epsilon)x + \lambda y, \eta' = (\lambda + \epsilon)\xi + \lambda\eta, \end{aligned} \tag{51}$$

where λ is an integer to be chosen. From this we obtain the inverse transformation

$$\begin{aligned} x &= \lambda x' + (\lambda + \epsilon)y', \xi = \lambda\xi' - (\lambda - \epsilon)\eta', \\ y &= (\lambda - \epsilon)x' + \lambda y', \eta = -(\lambda + \epsilon)\xi' + \lambda\eta'. \end{aligned} \tag{52}$$

It is evident from (51) and (52) that $(\xi', \eta') = 1$, and that both x' and y' are odd for all choices of λ. Also $\xi'x' + \eta'y' = n$. It remains to show that we can choose λ so that ξ', η', x' and y' are all positive. For x' and y' to be positive we must have

$$\epsilon y < \lambda(x - y) < \epsilon x,$$

i.e.

$$0 < \frac{\min(x, y)}{|x - y|} < \lambda < \frac{\max(x, y)}{|x - y|}. \tag{53}$$

The two numbers on the left and right of λ in this inequality differ by unity, and neither is an integer, since $|x - y|$ is even and both x and y are odd. Hence there exists one integral value of λ and one only for which x' and y' are positive. By (53), $\lambda \gtrless 1$, and hence ξ' and η' are positive. It follows that the new solution belongs to $S(n)$. By (51)

$$x' + y' = \epsilon(x - y) = |x - y|, \tag{54}$$

and

$$\xi' - \eta' = -\epsilon(\xi + \eta). \tag{55}$$

Also

$$|x - y| = x' + y' > |x' - y'|, \tag{56}$$

and

$$|\xi' - \eta'| = \xi + \eta > |\xi - \eta|. \tag{57}$$

It follows from (55) that

$$\epsilon = -\operatorname{sgn}(\xi' - \eta').$$

If $x \equiv y$ (mod4),

$$x' \equiv -\epsilon y \pmod 4, \quad y' \equiv \epsilon x \pmod 4,$$

so that

$$\chi(x') - \chi(y') = -\epsilon\{\chi(x) + \chi(y)\}. \tag{58}$$

If $x \equiv -y$ (mod 4), $\chi(x) + \chi(y) = 0$, and

$$x' \equiv (2\lambda + \epsilon)x \pmod 4, \quad y' \equiv -(2\lambda - \epsilon)x \pmod 4,$$

so that

$$\chi(x') = (-1)^{\lambda}\epsilon\chi(x), \quad \chi(y') = (-1)^{\lambda}\epsilon\chi(y),$$

and therefore (58) holds in this case too.

If $x' \neq y'$, we can continue this process and arrive at a new solution, and so on. By (56), we shall eventually obtain a solution for which $x'' = y''$.

We can also generate new solutions by proceeding in the opposite direction. Suppose that (x', y', ξ', η') is a solution for which $|\xi' - \eta'| > \delta_n$. Then we obtain a new solution (x, y, ξ, η) by the transformation

$$\begin{aligned} x &= \mu x' + (\mu + \epsilon')y', \ \xi = \mu\xi' - (\mu - \epsilon')\eta', \\ y &= (\mu - \epsilon')x' - \mu y', \ \eta = -(\mu + \epsilon')\xi' + \mu\eta', \end{aligned} \tag{59}$$

where

$$\epsilon' = -\operatorname{sgn}(\xi' - \eta').$$

Clearly x and y are both odd, and if we solve for ξ' and η' in terms of ξ' and η' we can show that $(\xi, \eta) = 1$. Also $\xi x + \eta y = n$. It remains to show that μ can be chosen so that ξ, η, x and y are all positive. For ξ and η to be positive we must have

$$\epsilon'\xi' < \mu(\eta' - \xi') < \epsilon'\eta',$$

i.e.

$$0 < \frac{\min(\xi', \eta')}{|\xi' - \eta'|} < \mu < \frac{\max(\xi', \eta')}{|\xi' - \eta'|}. \tag{60}$$

The two numbers on either side of μ in (60) differ by unity, and neither is an integer since $|\xi' - \eta'| > 1$. This is obvious if n is odd; if n is even ξ' and η' are both odd, so that $\xi' - \eta'$ is even, and hence $|\xi' - \eta'| \geq 2$. Hence there exists one integral value of μ and only one for which ξ and η are both positive. By (10), μ is positive, and hence x and y are also positive, and it follows that the solution (x, y, ξ, η) belongs to $S(n)$. By (59),

$$x - y = -\epsilon'(x' + y'), \tag{61}$$

and

$$\xi + \eta = -\epsilon'(\xi' - \eta') = |\xi' - \eta'|. \tag{62}$$

Hence

$$\epsilon' = \operatorname{sgn}(x - y).$$

Thus (61) and (62) are identical with (54) and (55) with ϵ' in place of ϵ. Formulae (56), (57), and (58) may be deduced in the same manner.

If $|\xi - \eta| \neq \delta_n$ we can continue the process and derive a new solution, and so on. By (57), we shall eventually obtain a solution $(x^*, y^*, \xi^*, \eta^*)$ for which $|\xi^* - \eta^*| = \delta_n$.

Now $\epsilon' = \epsilon$ since each is equal to $-\operatorname{sgn}(\xi' - \eta')$, and on comparing equations (52) and (59) we see that we must have $\lambda = \mu$, since both are uniquely determined. Thus to each solution (x, y, ξ, η) of $S(n)$ we can assign a unique successor, if $x \neq y$, and a unique predecessor if $|\xi - \eta| \neq \delta_n$. Thus we have shown that every solution of $S(n)$ is a member of a unique sequence or chain of solutions, and that the members of any such chain have the properties (44)–(49). Since $|\xi_1 - \eta_1| = \delta_n$ for the first member, and $x_q = y_q$ for the last member of each chain, it follows that each chain corresponds to a unique solution of $C(n)$ and to a unique solution of $C'(n)$. There are therefore $c(n)$ chains. This completes the proof of Theorem 1. It may be remarked that the numbeı q is not necessarily the same for different chains.

Let $\psi_0(u, v)$ and $\psi_1(u, v)$ be any two functions of the integers u and v, such that ψ_0 is an even function of both u and v, and ψ_1 is an even function of u and an odd function of v. Then, by Theorem 1, for two successive members of a chain, indexed by m and $m + 1$,

$$\psi_0(x_m - y_m, \xi_m + \eta_m) = \psi_0(x_{m+1} + y_{m+1}, \xi_{m+1} - \eta_{m+1}) \tag{63}$$

and

$$\begin{aligned}\psi_1(x_m - y_m, \xi_m + \eta_m)\{\chi(x_m) + \chi(y_m)\} \\ = \psi_1(x_{m+1} + y_{m+1}, \xi_{m+1} - \eta_{m+1})\{\chi(x_{m+1}) - \chi(y_{m+1})\}.\end{aligned} \tag{64}$$

Put

$$f_0(x, y, \xi, \eta) = \psi_0(x + y, \xi - \eta) - \psi_0(x - y, \xi + \eta), \tag{65}$$

and

$$\begin{aligned} &f_1(x, y, \xi, \eta) \\ &\quad = \psi_1(x + y, \xi - \eta)\{\chi(x) - \chi(y)\} - \psi_1(x - y, \xi + \eta)\{\chi(x) + \chi(y)\}. \end{aligned} \tag{66}$$

Then we have, by (63) and (64), for any chain of more than one member,

$$\sum_{m=1}^{q} f_0(x_m, y_m, \xi_m, \eta_m) = \psi_0(x_1 + y_1, \xi_1 - \eta_1) - \psi_0(0, \xi_q + \eta_q), \tag{67}$$

and

$$\begin{aligned} &\sum_{m=1}^{q} f_1(x_m, y_m, \xi_m, \eta_m) \\ &\quad = \psi_1(x_1 + y_1, \xi_1 - \eta_1)\{\chi(x_1) - \chi(y_1)\} - 2\psi_1(0, \xi_q - \eta_q)\chi(x_q). \end{aligned} \tag{68}$$

Formulae (67) and (68) also hold for a chain containing only one solution. It may be noted that it is possible to prove results of the same type for functions $\psi(u, v)$ that are odd functions of u, but such results have no application since the sums over $S(n)$ of the corresponding functions f vanish.

Theorem 2 *If $f_0(x, y, \xi, \eta)$ and $f_1(x, y, \xi, \eta)$ are defined as in* (65) *and* (66), *we have (i) if n is odd,*

$$\sum_{S(n)} f_0(x, y, \xi, \eta) = 2 \sum_{t=1}^{(n-1)/2} \psi_0(2t, 1) - \sum_{\delta | n} \phi(\delta)\psi_0(0, \delta) + \psi_0(0, 1), \tag{69}$$

$$\begin{aligned} \sum_{S(n)} f_1(x, y, \xi, \eta) = 4\chi(n) &\sum_{u=1}^{\frac{1}{4}\{n-2+\chi(n)\}} \psi_1(4u, 1) \\ &- 2\sum_{\delta | n} \chi(\delta)\psi_1(0, \delta')\phi(\delta') + 2\chi(n)\psi_1(0, 1), \end{aligned} \tag{70}$$

and (ii) if n is even

$$\sum_{S(n)} f_0(x, y, \xi, \eta) = \frac{1}{2}\psi_0(n, 0) - \sum_{\delta | n} \phi(\delta')\psi_0(0, \delta), \tag{71}$$

$$\sum_{S(n)} f_1(x, y, \xi, \eta) = -2\sum_{\delta | n} \chi(\delta)\phi(\delta')\psi_1(0, \delta'). \tag{72}$$

By (67)

$$\sum_{S(n)} f_0(x, y, \xi, \eta) = \sum_{C(n)} \psi_0(x + y, \xi - \eta) - \sum_{C'(n)} \psi_0(0, \xi + \eta),$$

and (69) and (71) follow from Lemmas 1, 2, 3 and 5. Also, by (68),

$$\sum_{S(n)} f_1(x, y, \xi, \eta) = \sum_{C(n)} \psi_1(x + y, \xi - \eta)\{\chi(x) - \chi(y)\} - 2\sum_{C'(n)} \psi_1(0, \xi + \eta)\chi(n).$$

By Lemma 5,

$$2\sum_{C(n)} \psi_1(0, \xi + \eta)\chi(n) = 2 \sum_{\delta | n, \delta < n} \chi(\delta)\phi(\delta')\psi_1(0, \delta'). \tag{73}$$

To evaluate

$$\sum_{C(n)} \psi_1(x + y, \xi - \eta)\{\chi(x) - \chi(y)\}$$

we observe that $\chi(x) - \chi(y) = 0$ unless $x + y \equiv 0 \pmod 4$, so that it is only necessary to consider values of x and y which satisfy this congruence.

Suppose that n is odd. Then

$$n = \xi(x + y) + (\eta - \xi)y \equiv (\eta - \xi)y \pmod 4,$$

and therefore

$$\chi(x) - \chi(y) = -2\chi(y) = 2\chi(n)\chi(\xi - n).$$

Thus

$$\sum_{C(n)} \psi_1(x + y, \xi - \eta)\{\chi(x) - \chi(y)\} = 2\chi(n) \sum_{1 \le u < n/4} \{\psi_1(4u, 1) - \psi_1(4u, -1)\}$$

$$= 4\chi(n) \sum_{u=1}^{\frac{1}{4}\{n-2+\chi(n)\}} \psi_1(4u, 1). \tag{74}$$

Finally, if n is even,

$$\sum_{C(n)} \psi_1(x + y, \xi - \eta)\{\chi(x) - \chi(y)\} = \psi_1(n, 0) \sum_{x+y=n} \{\chi(x) - \chi(y)\} = 0, \tag{75}$$

by Lemma 1.

Equations (70) and (72) now follow from (73), (74) and (75).

4 Two Squares

We assume (1). As stated in §2, it may be proved by elementary methods. Equations (7) and (8) are particular cases of (1).

5 Four Squares

Take

$$\psi_0(u, v) = \frac{1}{2}(1 - \cos \pi u/2),$$

so that, by (65),

$$\begin{aligned} f(x, y) &= f_0(x, y, \xi, \eta) \\ &= -\frac{1}{2}\left\{\cos\frac{\pi}{2}(x+y) - \cos\frac{\pi}{2}(x-y)\right\} = \sin\frac{\pi}{2}x \sin\frac{\pi}{2}y = \chi(x)\chi(y). \end{aligned}$$

By Theorem 2, if n is odd,

$$\sum_{S(n)} f(x, y) = \sum_{t=1}^{(n-1)/2} \{1 - (-1)^t\} = \frac{1}{2}\{n - \chi(n)\}.$$

If n is even,

$$\sum_{S(n)} f(x, y) = \frac{1}{4}n\left(1 - \cos\frac{\pi}{2}n\right) = \frac{1}{4}n\{1 - (-1)^{n/2}\}.$$

Hence, by (34),

$$\sum_{T(2n+1)} f(x, y) = \frac{1}{2}\sum_{\delta|2n+1}\{\delta - \chi(\delta)\} = \frac{1}{2}\{\Delta_1(2n+1) - E_0(2n+1)\}, \tag{76}$$

and

$$\begin{aligned} \sum_{T(2n)} f(x, y) &= \sum_{d|2n}\sum_{S(d)} f(x, y) = \frac{1}{2}\sum_{\delta|n}\{\delta - \chi(\delta)\} + \frac{1}{2}\sum_{d|n} d\{1 - (-1)^d\} \\ &= \frac{3}{2}\Delta_1(n) - \frac{1}{2}E_0(n). \end{aligned} \tag{77}$$

It follows from (76) and (77) that

$$\sum_{T(n)} f(x, y) = \frac{1}{2}\{(2 + (-1)^n)\Delta_1(n) - E_0(n)\}. \tag{78}$$

Also, by (35),

$$\sum_{T_1(2n)} f(x, y) = \frac{1}{2}\sum_{\delta|n}\delta'\{1 - (-1)^{\delta'}\} = \frac{1}{2}\{1 - (-1)^n\}\Delta_1(n). \tag{79}$$

Now, by (32) and (78),

$$\begin{aligned} R_4(n) &= \sum_{m=1}^{n-1} R_2(m)R_2(n-m) + 2R_2(n) = 16\sum_{m=1}^{n-1} E_0(m)E_0(n-m) + 8E_0(n) \\ &= 16\sum_{T(n)} f(x, y) + 8E_0(n) = 8\{2 + (-1)^n\}\Delta_1(n). \end{aligned}$$

Also, by (7), (31), (33) and (79),

$$\begin{aligned} R_{4,0}(8n+4) &= \sum_{m=0}^{n} R_{2,0}(4m+2)R_{2,0}(8n-4m+2) \\ &= 16\sum_{m=0}^{2n} E_0(2m+1)E_0(4n-2m+1) \\ &= 16\sum_{T_1(4n+2)} f(x,y) = 16\Delta_1(2n+1), \end{aligned}$$

and, by (7), (8), (29), (31), (32) and (76),

$$\begin{aligned} R_{3,1}(4n+3) &= 2\sum_{m=0}^{2n} R_{2,0}(2m+2)R_{1,1}(4n-2m+1) \\ &= 32\sum_{m=0}^{2n} E_0(2m+2)E_0(4n-2m+1) \\ &= 16\sum_{\mu=1}^{4n+2} E_0(\mu)E_0(4n+3-\mu) \\ &= 16\sum_{T(4n+3)} f(x,y) = 8\Delta_1(4n+3). \end{aligned}$$

In a similar manner it can be shown that

$$R_{2,2}(4n+2) = 24\sum_{T_1(4n+2)} f(x,y) = 24\Delta_1(2n+1),$$

and (by using the fact that $R_{0,2}(4m) = R_2(m)$)

$$R_{1,3}(4n+1) = 16\sum_{T(4n+1)} f(x,y) + 2R_{1,1}(4n+1) = 8\Delta_1(4n+1).$$

6 Six Squares

Take $\psi_1(u,v) = \frac{1}{2}v$. Then, by (66),

$$\begin{aligned} f(x,y,\xi,\eta) &= -\frac{1}{2}(\xi-\eta)\{\chi(x)-\chi(y)\} + \frac{1}{2}(\xi+\eta)\{\chi(x)+\chi(y)\} \\ &= \xi\chi(y) + \eta\chi(x), \end{aligned} \tag{80}$$

and, by Theorem 2,

$$\sum_{S(n)} f_1(x,y,\xi,\eta) = \sum_{\delta|n} \delta'\phi(\delta')\chi(\delta),$$

when n is even; and therefore, since (36) is satisfied with $\nu = 1$, we have, by (38) and Lemma 4,

$$\begin{aligned}\sum_{T_1(2n)} f_1(x,y,a,b) &= \sum_{\delta|2n}\delta \sum_{S(2n/\delta)} f_1(x,y,\xi,\eta)\\ &= \sum_{\delta|n}\delta \sum_{\delta_1|\frac{n}{\delta}} \frac{2n}{\delta\delta_1}\phi\left(\frac{2n}{\delta\delta_1}\right)\chi(\delta_1)\\ &= 2\sum_{\delta_1|n}\frac{n}{\delta_1}\chi(\delta_1)\sum_{\delta|n/\delta_1}\phi\left(\frac{2n}{\delta\delta_1}\right)\\ &= 2\sum_{\delta_1|n}\chi(\delta_1)\left(\frac{n}{\delta_1}\right)^2 = 2E_2'(n). \qquad (81)\end{aligned}$$

Write

$$U_0(n) = \sum_{ax+by=n,2|ax} x\chi(y), \qquad (82)$$

$$U_1(n) = \sum_{ax+by=n,2\nmid ax} x\chi(y), \qquad (83)$$

and

$$U(n) = U_0(n) + U_1(n). \qquad (84)$$

By (80), (81) and (82),

$$\begin{aligned}U_1(2n) = \sum_{T_1(2n)} x\chi(y) &= \sum_{T_1(2n)} a\chi(y) = \frac{1}{2}\sum_{T_1(2n)}\{a\chi(y)+b\chi(x)\}\\ &= \frac{1}{2}\sum_{T_1(2n)} f_1(x,y,a,b)\\ &= E_2'(n), \qquad (85)\end{aligned}$$

since a and b are odd as well as x and y, for members of $T_1(2n)$.

With the help of (85) we can evaluate $U_0(n)$ and $U_1(n)$ in the general case. For, by (1), (2), (7)–(12) and (33),

$$\begin{aligned}R_{4,2}(4n) &= \frac{5}{2}\sum_{m=0}^{n-1} R_{2,2}(4m+2)R_{2,0}(4n-4m-2)\\ &= 240\sum_{m=0}^{n-1}\Delta_1(2m+1)E_0(2n-2m-2)\\ &= 240U_1(2n), \qquad (86)\end{aligned}$$

$$\begin{aligned}R_{4,2}(8n+4) &= 15\sum_{m=0}^{n} R_{4,0}(8m+4)R_{0,2}(8n-8m)\\ &= 960U_1(2n+1) + 240\Delta_1(2n+1) \qquad (87)\end{aligned}$$

$$\begin{aligned} R_{2,4}(4n+2) &= \frac{5}{2}\sum_{m=0}^{n} R_{2,2}(4m+2)R_{0,2}(4n-4m) \\ &= 240U_1(2n+1)+60\Delta_1(2n+1), \end{aligned} \tag{88}$$

$$\begin{aligned} R_{2,4}(8n+2) &= 15\sum_{m=0}^{n} R_{2,0}(8m+2)R_{0,4}(8n-8m) \\ &= 1440\sum_{m=0}^{n-1} E_0(4m+1)\Delta_1(4n-4m)+60E_0(4n+1) \\ &= 1440\sum_{m=0}^{2n-1} E_0(2n+1)\Delta_1(4n-2m)+60E_0(4n+1) \\ &= 1440U_0(4n+1)+60E_0(4n+1), \end{aligned} \tag{89}$$

$$\begin{aligned} R_{2,4}(8n+6) &= 15\sum_{m=0}^{n} R_{2,0}(8m+2)R_{0,4}(8n-8m+4) \\ &= 480\sum_{m=0}^{n} E_0(4m+1)\Delta_1(4n-4m+2) \\ &= 480\sum_{m=0}^{2n} E_0(2n+1)\Delta_1(4n-2m+2) \\ &= 480U_0(4n+3), \end{aligned} \tag{90}$$

$$\begin{aligned} R_{6,0}(8n+6) &= \sum_{m=0}^{n} R_{4,0}(8m+4)R_{2,0}(8n-8m+2) \\ &= 64\sum_{m=0}^{n} E_0(4n-4m+1)\Delta_1(4m+2) \\ &= 64\sum_{m=0}^{2n} E_0(4n-2m+1)\Delta_1(2m+2) \\ &= 64U_0(4n+3), \end{aligned} \tag{91}$$

$$\begin{aligned} R_{3,3}(4n+3) &= \frac{5}{2}\sum_{m=0}^{n}\{R_{3,1}(4m+3)R_{0,2}(4n-4m) \\ &\quad + R_{1,3}(4m+1)R_{2,0}(4n-4m+2)\} \\ &= 80\sum_{m=0}^{2n} \Delta_1(2m+1)E_0(4n-2m+2)+20\Delta_1(4n+3) \\ &= 80U_1(4n+3)+20\Delta_1(4n+3), \end{aligned} \tag{92}$$

and

$$\begin{aligned} R_6(n) &= \sum_{m=0}^{n} R_4(m)R_2(n-m) \\ &= 96U(n) - 64U_1(n) + 4E_0(n) + 8\{2+(-1)^n\}\Delta_1(n). \end{aligned} \tag{93}$$

It follows from (85), (86) and (87) that

$$\begin{aligned} U_1(2n+1) &= \frac{1}{4}\{U_1(4n+2) - \Delta_1(2n+1)\} \\ &= \frac{1}{4}\{E_2'(2n+1) - \Delta_1(2n+1)\}. \end{aligned} \tag{94}$$

This formula may be combined with (85) to give

$$U_1(n) = \frac{1}{4}\left\{E_2'(n) - \frac{1}{2}(1-(-1)^n)\Delta_1(n)\right\}. \tag{95}$$

By (88), (89) and (90),

$$\begin{aligned} 24U_0(4n+1) &= 4U_1(4n+1) - E_0(4n+1) + \Delta_1(4n+1) \\ &= E_2'(4n+1) - E_0(4n+1), \end{aligned}$$

and

$$24U_0(4n+3) = 12U_1(4n+3) + 3\Delta_1(4n+3) = 3E_2'(4n+3),$$

which combine to give, by (30) and (31),

$$U_0(2n+1) = \frac{1}{24}\{2E_2'(2n+1) - E_2(2n+1) - E_0(2n+1)\}. \tag{96}$$

From (94) and (96) it follows that

$$U(n) = \frac{1}{24}\{8E_2'(n) - E_2(n) - E_0(n) - 6\Delta_1(n)\}, \tag{97}$$

if n is odd, and we shall show that this holds also when n is even. For we have, if $n = 2^\alpha m$ where m is odd and α is positive,

$$\begin{aligned} U(n) &= \sum_{\beta=1}^{\alpha} U_1(2^\beta m) + U(m) \\ &= \sum_{\beta=1}^{\alpha} E_2'(2^\beta m) + U(m) \\ &= \frac{1}{3}(2^{2\alpha}-1)E_2'(m) + \frac{1}{24}\{8E_2'(m) - E_2(m) - E_0(m) - 6\Delta_1(m)\} \\ &= \frac{1}{3}E_2'(n) - \frac{1}{24}\{E_2(n) + E_0(n) + 6\Delta_1(n)\}, \end{aligned}$$

by (28) and (29), and this is the same as (97). It follows from (83) and (95) that

$$\begin{aligned} U_0(2n) &= U(2n) - U_1(2n) \\ &= \frac{1}{24}\{8E_2'(n) - E_2(n) - E_0(n) - 6\Delta_1(n)\} = U(n). \end{aligned}$$

This may also be deduced immediately from (82), (83) and (84).

Formulae (3), (13), (14), (15) and (16) now follow from (86), (88), (91), (92) and (93), with the help of (85), (94), (95), (96) and (97).

7 Eight Squares

Take $\psi_0(u, v) = \frac{1}{4}u^2$. Then, by (65),

$$f_0(x, y, \xi, \eta) = f(x, y) = xy.$$

It follows from Theorem 2 that

$$\sum_{S(n)} f(x, y) = 2 \sum_{t=1}^{(n-1)/2} t^2 = \frac{1}{12} n(n^2 - 1),$$

when n is odd, and

$$\sum_{S(n)} f(x, y) = \frac{1}{8} n^3$$

when n is even.

Therefore, if n is odd, by (34),

$$\sum_{T(n)} f(x, y) = \frac{1}{12} \sum_{\delta|n} \delta'(\delta'^2 - 1) = \frac{1}{12}\{\Delta_3(n) - \Delta_1(n)\}. \tag{98}$$

Suppose that $n = 2^\alpha m$, where m is odd and α is positive. Then

$$\begin{aligned} \sum_{T(n)} f(x, y) &= \sum_{\delta|n} \left\{ \frac{1}{12}\delta(\delta^2 - 1) + \frac{1}{8}\delta^3(2^3 + 2^6 + \cdots + 2^{3\alpha}) \right\} \\ &= \sum_{\delta|n} \left\{ \frac{1}{12}\delta(\delta^3 - 1) + \frac{1}{7}(2^{3\alpha} - 1)\delta^3 \right\} \\ &= \frac{1}{84}\{12\Delta_3'(n) - 5\Delta_3(n) - 7\Delta_1(n)\}, \end{aligned} \tag{99}$$

and, by (98) this holds also when n is odd. Also, by (35),

$$\sum_{T_1(n)} f(x, y) = \Delta_3'(n). \tag{100}$$

We now apply these results. We have

$$R_8(n) = 2R_4(n) + \sum_{m=1}^{n-1} R_4(m)R_4(n-m). \tag{101}$$

If n is odd one of the numbers $m, n-m$ is odd and the other is even, so that, by (2),

$$\begin{aligned} R_8(n) &= 16\Delta_1(n) + 192\sum_{m=1}^{n-1} \Delta_1(m)\Delta_1(n-m) \\ &= 16\Delta_1(n) + 192\sum_{T(n)} f(x, y) \\ &= 16\Delta_3(n). \end{aligned}$$

Also, by (2), (28), (99), (100), and (101),

$$\begin{aligned} R_8(2n) &= 48\Delta_1(n) + 64\sum_{m=1}^{n-1} \Delta_1(2m-1)\Delta_1(2n-2m+1) \\ &\quad + 576\sum_{m=1}^{n-1} \Delta_1(2m)\Delta_1(2n-2m) \\ &= 48\Delta_1(n) + 64\sum_{T_1(2n)} f(x, y) + 576\sum_{T(n)} f(x, y) \\ &= \frac{16}{7}\{8\Delta_3'(2n) - 15\Delta_3(2n)\}. \end{aligned}$$

Formula (4) follows from the last two results.

By (9) and (100),

$$\begin{aligned} R_{8,0}(8n) &= \sum_{m=0}^{n-1} R_{4,0}(8m+4)R_{4,0}(8n-8m-4) \\ &= 256\sum_{m=0}^{n-1} \Delta_1(2m+1)\Delta_1(2n-2m-1) \\ &= 256\sum_{T_1(2n)} f(x, y) = 256\Delta_3'(n), \end{aligned}$$

which is formula (17). By (11) and (33),

$$\begin{aligned} R_{4,4}(4n) &= \frac{35}{18}\sum_{m=0}^{n-1} R_{2,2}(4m+2)R_{2,2}(4n-4m-2) \\ &= 1120\sum_{m=0}^{n-1} \Delta_1(2m+1)\Delta_1(2n-2m-1) \\ &= 1120\sum_{T_1(2n)} f(x,y) = 1120\Delta_3'(n), \end{aligned}$$

which is (18).

We conclude by proving a formula which we shall require in the next section. By (33), (10) and (12),

$$\begin{aligned} R_{6,2}(4n+2) &= \frac{7}{4}\sum_{m=0}^{n-1} R_{3,1}(4m+3)R_{3,1}(4n-4m-1) \\ &= 112\sum_{m=0}^{n-1} \Delta_1(4m+3)\Delta_1(4n-4m-1), \end{aligned}$$

and

$$\begin{aligned} R_{2,6}(4n+2) &= \frac{7}{4}\sum_{m=0}^{n-1} R_{1,3}(4m+1)R_{1,3}(4n-4m+1) \\ &= 112\sum_{m=0}^{n-1} \Delta_1(4m+1)\Delta_1(4n-4m+1). \end{aligned}$$

Hence, by (100),

$$\begin{aligned} R_{6,2}(4n+2) + R_{2,6}(4n+2) &= 112\sum_{m=0}^{2n} \Delta_1(2m+1)\Delta_1(4n-2m+1) \\ &= 112\sum_{T_1(4n+2)} f(x,y) \\ &= 112\Delta_3(2n+1). \end{aligned} \tag{102}$$

8 Ten Squares

Take $\psi_1(u,v) = -\frac{1}{4}v^3$, so that, by (66),

$$f_1(x,y,a,b) = \frac{1}{2}\{a^3\chi(y) + b^3\chi(a)\} + \frac{3}{2}ab\,\{a\chi(x) + b\chi(y)\}. \tag{103}$$

It follows from Theorem 2 and (38) that

$$\sum_{S(2n)} f_1(x, y, \xi, \eta) = 4\sum_{\delta|n} \chi(\delta)\phi\left(\frac{2n}{\delta}\right)\left(\frac{n}{\delta}\right)^3,$$

and

$$\begin{aligned}
\sum_{T_1(2n)} f_1(x, y, a, b) &= \sum_{\delta_1|n}\delta_1^3 \sum_{S(2n/\delta_1)} f_1(x, y, \xi, \eta) \\
&= 4\sum_{\delta_1|n}\delta_1^3 \sum_{\delta|n/\delta_1} \chi(\delta)\phi\left(\frac{2n}{\delta\delta_1}\right)\left(\frac{n}{\delta\delta_1}\right)^3 \\
&= 4\sum_{\delta|n}\left(\frac{n}{\delta}\right)^3 \chi(\delta) \sum_{\delta_1|n/\delta} \phi\left(\frac{2n}{\delta\delta_1}\right) \\
&= 4\sum_{\delta|n}(\delta')^4\chi(\delta) \\
&= 4E_4'(n). \qquad (104)
\end{aligned}$$

On the other hand, by (103),

$$\begin{aligned}
\sum_{T_1(n)} f_1(x, y, a, b) &= \sum_{m=0}^{n-1}\{\Delta_3(2m+1)E_0(2n-2m-1) \\
&\quad + 3E_2'(2m+1)\Delta_1(2n-2m-1)\}. \qquad (105)
\end{aligned}$$

Put $N = 2^\alpha N_1$, where $\alpha \geq 0$ and $N_1 \equiv 3 \pmod 4$. Write

$$V_1(N) = \sum E_2(4m+1)\Delta_1(N-4m-1), \qquad (106)$$

$$V_3(N) = \sum E_2(4m+3)\Delta_1(N-4m-3), \qquad (107)$$

$$V_0(N) = \sum E_2(2m)\Delta_1(N-2m), \qquad (108)$$

$$V_0'(N) = \sum E_2'(2m)\Delta_1(N-2m), \qquad (109)$$

$$W(N) = \sum \Delta_3(2m+1)E_0(2N-2m-1), \qquad (110)$$

where in each case the summation is extended over all m for which the arguments of the functions Δ_1, Δ_3, E_0, E_2 and E_2' are positive.

It follows from (104), (105), (106), (107) and (110) that

$$W(N) + 3\{V_1(2N) - V_3(2N)\} = 4E_4'(N). \qquad (111)$$

By (33), (102) and (110),

$$\begin{aligned}
3R_{4,6}(4N) + 14R_{8,2}(4N) &= \frac{45}{2}\sum_{m=0}^{N-1}\{R_{6,2}(4m+2) + R_{2,6}(4m+2)\} \\
&\quad \times R_{2,0}(4N-4m-2) = 10080W(N). \qquad (112)
\end{aligned}$$

Also, by (2), (3) and (32),

$$\begin{aligned} R_{10}(N) &= \sum_{m=1}^{N-1} R_6(m)R_4(N-m) + R_6(N) + R_4(N) \\ &= 32\sum_{m=1}^{N-1}\{4E_2'(m) - E_2(m)\}\{2+(-1)^{N-m}\}\Delta_1(N-m) + R_6(N) + R_4(N). \end{aligned}$$

Hence we have, if $\alpha = 0$,

$$\begin{aligned} R_{10}(N) &= 288V_1(N) - 480V_3(N) + 128V_0'(N) - 32V_0(N) \\ &\quad + 8\Delta_1(N) + 20E_2'(N), \end{aligned} \tag{113}$$

and, if $\alpha > 0$,

$$\begin{aligned} R_{10}(N) &= 96V_1(N) - 160V_3(N) + 384V_0'(N) - 96V_0(N) \\ &\quad + 24\Delta_1(N) + 16E_2'(N) - 4E_2(N). \end{aligned} \tag{114}$$

The following results may be proved in a similar manner by means of (33), and (1)–(4), (7)–(8). The group of four numbers after each formula indicates the values of (a, b, c, d) used in applying (33).

$$R_{6,4}(2N_1) = 3360V_0'(N_1) \quad (4, 2, 2, 2), \tag{115}$$

$$= 13440V_1(N_1) \quad (4, 0, 2, 4), \tag{116}$$

$$= -40320V_3(N_1) + 1680E_2'(N_1) \quad (6, 0, 0, 4), \tag{117}$$

$$R_{2,8}(2N_1) = 1440\{V_1(N_1) - 3V_3(N_1)\} + 180E_2'(N_1)(2, 4, 0, 4), \tag{118}$$

$$R_{8,2}(4N) = -1440V_3(2N) \quad (6, 0, 2, 2), \tag{119}$$

$$= 11520V_0'(N)(\alpha = 0) \quad (4, 0, 4, 2), \tag{120}$$

$$= 11520\{V_1(N) - V_3(N)\}(\alpha > 0) \quad (4, 0, 4, 2), \tag{121}$$

$$R_{4,6}(4N) = 3360\{V_1(2N) - V_3(2N)\} \quad (2, 4, 2, 2), \tag{122}$$

$$R_{4,6}(N) = 13440\{4V_0'(N) - V_1(N)\} + 3360\Delta_1(N)(\alpha = 0) \quad (4, 0, 0, 6), \tag{123}$$

$$\begin{aligned} &= 26880\{V_0'(N) + 3V_1(N) - 3V_3(N)\} + 3360E_2'(N) \\ &\quad (\alpha = 0)(4, 2, 0, 4), \end{aligned} \tag{124}$$

$$\begin{aligned} &= 26880\{3V_0'(N) + V_1(N) - V_3(N)\} + 3360E_2'(N) \\ &\quad (\alpha > 0)(4, 2, 0, 4). \end{aligned} \tag{125}$$

By (112), (119) and (122),

$$V_1(2N) - 3V_3(2N) = W(N), \tag{126}$$

and therefore, by (111),

$$2V_1(2N) - 3V_3(2N) = 2E_4'(N). \tag{127}$$

We consider first the case $\alpha = 0$. It follows from (115), (116) and (117) that

$$V_1(N_1) = \frac{1}{4}V_0'(N_1), \quad V_3(N_1) = -\frac{1}{24}\{2V_0'(N_1) - E_2'(N_1)\}, \tag{128}$$

and, from (122), (123), (124) and (128), that

$$V_0(N_1) = \frac{1}{4}\Delta_1(N_1), \quad V_1(2N_1) - V_3(2N_1) = 16V_0'(N_1). \tag{129}$$

Thus we have, in addition to (115) and (120),

$$\begin{aligned} R_{4,6}(4N_1) = 53760V_0'(N_1), R_{2,8}(2N_1) &= 728V_0'(N_1), R_{10}(N_1) \\ &= 240V_0'(N_1), \end{aligned} \tag{130}$$

and therefore, by (112),

$$W(N_1) = 32V_0'(N_1). \tag{131}$$

It follows from (111) that

$$V_0'(N_1) = \frac{1}{20}E_4'(N_1). \tag{132}$$

Thus we have, by (127), (128), (129), (131) and (132),

$$V_1(N_1) = \frac{1}{80}E_4'(N_1), V_3(N_1) = -\frac{1}{240}\{E_4'(N_1) - 10E_2'(N_1)\}, \tag{133}$$

$$W_1(N_1) = \frac{8}{5}E_2'(N_1), V_1(2N_1) = \frac{1}{41}E_4'(2N_1), \tag{134}$$

$$V_3(2N_1) = -\frac{1}{40}E_4'(2N_1). \tag{135}$$

The functions $V_0(N_1)$ and $V_0'(N_1)$ are given by (129) and (132).

We now suppose that $\alpha > 0$. By (119) and (121),

$$V_3(2N) + 8V_1(N) - 8V_3(N) = 0. \tag{136}$$

By (127) with $\frac{1}{2}N$ in place of N,

$$2V_1(N) - 3V_3(N) = \frac{1}{8}E_4'(N). \tag{137}$$

Eliminating $V_1(N)$ between (136) and (137), we obtain

$$V_3(2N) + 4V_3(N) + \frac{1}{2}E_4'(N) = 0,$$

which may be set in the following form:

$$V_3(2N) + \frac{1}{40}E_4'(2N) = -4\left\{V_3(N) + \frac{1}{40}E_4'(N)\right\}. \tag{138}$$

It follows from (135) by repeated application of (138) that

$$V_3(N) = -\frac{1}{40}E_4'(N). \tag{139}$$

Hence, by (126) and (137),

$$V_1(N) = \frac{1}{40}E_4'(N),\ W(N) = \frac{8}{5}E_4'(N). \tag{140}$$

By (122), (125), (139) and (140),

$$V_0'(N) = \frac{1}{60}E_4'(N) - \frac{1}{24}E_2'(N). \tag{141}$$

It remains to evaluate $V_0(N)$ when N is even. By (108), (139) and (140),

$$\begin{aligned} V_0(2N) &= \sum_{m=1}^{N-1} E_2(2m)\Delta_1(2N-2m) = \sum_{m=1}^{N-1} E_2(m)\Delta_1(N-m) \\ &= V_0(N) + V_1(N) + V_3(N) = V_0(N). \end{aligned} \tag{142}$$

Similarly, by (129) and (133),

$$\begin{aligned} V_0(2N_1) &= V_0(N_1) + V_1(N_1) + V_3(N_1) \\ &= \frac{1}{4}\Delta_1(N_1) - \frac{1}{120}E_4(N_1) - \frac{1}{24}E_2(N_1). \end{aligned} \tag{143}$$

It follows from (143) by repeated application of (142) that

$$V_0(N) = \frac{1}{4}\Delta_1(N) - \frac{1}{120}E_4(N_1) - \frac{1}{24}E_2(N). \tag{144}$$

Formulae (5), (19) and (21) for $\alpha > 0$ now follow from (114), (119), (122), (139), (140), (141) and (144).

In the preceding analysis it has been assumed that $N_1 \equiv 1 \pmod 4$. It might be thought that formulae for the case $N_1 \equiv 3 \pmod 4$ could be obtained in a similar manner. This, however, is not the case. For if we set up the equations corresponding to equations (113)–(125) we find that we cannot eliminate the five functions (106)–(110) with the help of (111) and (112) and obtain formulae for the number of representations $R_{2\alpha,10-2\alpha}$. The reason for this is, as is well known, that in the general case it is necessary to introduce a new function which is not expressible in terms of divisor functions. This new function is Glaisher's function $\chi_4(n)$ which is defined by

$$\chi_4(n) = \sum (a+ib)^4,$$

where the summation is carried out over all representations of n as a^2+b^2. When n is not representable as a sum of two squares, e.g. when $n \equiv 3 \pmod 4$, $\chi_4(n)$ vanishes.

9 Twelve Squares

Take $\psi_0(u, v) = \frac{1}{16}u^4$, so that, by (65),

$$f(x, y) = f_0(x, y) = \frac{1}{2}(xy^3 + x^3y).$$

By Theorem 2, if n is odd,

$$\begin{aligned} \sum_{S(n)} f(x, y) &= 2 \sum_{t=1}^{(n-1)/2} t^4 \\ &= \frac{1}{240} n(n^2 - 1)(3n^2 - 7), \end{aligned} \tag{145}$$

and, if n is even,

$$\sum_{S(n)} f(x, y) = \frac{1}{32} n^5. \tag{146}$$

Write

$$Z(2n) = \sum_{m=1}^{2n-1} \Delta_1(m)\Delta_3(2n - m), \tag{147}$$

$$Z'(n) = \sum_{m=1}^{n-1} \Delta_1(m)\Delta_3'(n - m), \tag{148}$$

$$Z_1(2n) = \sum_{m=1}^{n} \Delta_1(2m - 1)\Delta_3(2n - 2m + 1), \tag{149}$$

It follows from these definitions and from (34), (35), (145) and (146) that

$$\begin{aligned} Z_1(2n) &= \sum_{T_1(2n)} f(x, y) = \sum_{\delta | 2n} \sum_{S(2n)/\delta} f(x, y) \\ &= \frac{1}{32} \Delta_5'(2n), \end{aligned} \tag{150}$$

and if $n = 2^\alpha m$, where m is odd and $\alpha \geq 0$,

$$\begin{aligned} Z(2n) &= \sum_{T_1(2n)} f(x, y) = \sum_{d|2n} \sum_{S(d)} f(x, y) = \sum_{\delta | n} \sum_{\beta=0}^{\alpha+1} \sum_{S(2^\beta \delta)} f(x, y) \\ &= \sum_{\delta | n} \left\{ \frac{1}{240} \delta(\delta^3 - 1)(3\delta^2 - 7) + \delta^5(1 + 2^5 + \cdots + 2^{5\alpha}) \right\} \\ &= \tfrac{1}{240} \{3\Delta_5(2n) - 10\Delta_3(2n) + 7\Delta_1(2n)\} \\ &\quad + \tfrac{1}{31} \{\Delta_5'(2n) - \Delta_5(2n)\}. \end{aligned} \tag{151}$$

We have, by (2), (4) and (32)

$$\begin{aligned} R_{12}(2n) &= R_4(2n) + R_8(2n) + \sum_{m=1}^{2n-1} R_4(m) R_8(2n-m) \\ &= R_4(2n) + R_8(2n) + 128 Z_1(2n) \\ &\quad + \tfrac{384}{7} \sum_{m=1}^{n-1} \Delta_1(2m)\{8\Delta_3'(2n-2m) - 15\Delta_3(2n-2m)\} \\ &= 24\Delta_1(2n) + \tfrac{16}{7}\{8\Delta_3'(2n) - 15\Delta_3(2n)\} \\ &\quad + 512 Z_1(2n) + \tfrac{384}{7}\{8Z'(2n) - 15Z(2n)\}. \end{aligned} \tag{152}$$

Similarly, by (33) and the formulae already proved, we obtain

$$R_{4,8}(8n) = 126720 Z_1(2n), \quad (4,0,0,8) \tag{153}$$

$$R_{8,4}(8n) = 126720 Z_1(2n), \quad (4,0,4,4) \tag{154}$$

$$= 3041280 Z'(n) + 126720\Delta_3'(n). \quad (8,0,0,4) \tag{155}$$

It follows from (150), (153) and (154) that

$$R_{4,8}(8n) = R_{8,4}(8n) = 3960\Delta_5'(2n). \tag{156}$$

Finally, by (155) and (156),

$$Z'(n) = \frac{1}{24}\{\Delta_5'(n) - \Delta_3'(n)\}, \tag{157}$$

and (6) follows from (150), (151), (152) and (157).

10 Concluding Remarks

If we examine the formulae that have been proved we notice certain relations connecting different types of representations, such as

$$5R_{6,0}(8n+6) = 2R_{3,3}(4n+3)$$

and

$$3R_{4,0}(8n+4) = 2R_{0,4}(8n+4).$$

These are particular cases of the following three theorems.

Theorem 3 *Let α, β, a, b and k be non-negative integers such that*

$$2k \equiv \alpha \pmod 4, \quad k \equiv a \pmod 4, \quad 0 \le k < 4,$$

and suppose that

$$R_{\alpha,\beta}(8n+2k) = A R_{a,b}(4n+k)$$

for all positive $4n+k$ *and fixed* A. *Then*

$$R_{\alpha+2m,\beta}(8n+2k+2m) = A_m R_{a+m,b+m}(4n+k+m) \tag{158}$$

for $m \geq 0$, *where*

$$A_m = 2^m A \frac{(\alpha+\beta+2m)!\alpha!(a+b)!(a+m)!(b+m)!}{(\alpha+\beta)!(\alpha+2m)!(a+b+2m)!a!}.$$

Theorem 4 *If* α, β, a, b *and* k *are non-negative integers such that*

$$k \equiv \alpha \equiv a \pmod 4, \quad 0 \leq k < 8,$$

and if

$$R_{\alpha,\beta}(8n+k) = B R_{a,b}(8n+k)$$

for all positive $8n+k$ *and fixed* B, *then*

$$R_{\alpha+2m,\beta}(8n+k+2m) = B_m R_{a+2m,b}(8n+k+2m)$$

for $m \geq 0$, *where*

$$B_m = B\frac{(\alpha+\beta+2m)!\alpha!(a+b)!(a+2m)!}{(\alpha+\beta)!(\alpha+2m)!(a+b+2m)!a!}.$$

Theorem 5 *If* α, β, a, b *and* k *are non-negative integers such that*

$$k \equiv \alpha \equiv a \pmod 4, \quad a \leq k < 4,$$

and if

$$R_{\alpha,\beta}(4n+k) = C R_{a,b}(4n+k)$$

for all positive $4n+k$ *and fixed* C, *then*

$$R_{\alpha+m,\beta+m}(4n+k+m) = C_m R_{a+m,b+m}(4n+k+m),$$

for $m \geq 0$, *where*

$$C_m = C\frac{(\alpha+\beta+2m)!\alpha!\beta!(a+b)!(a+m)!(b+m)!}{(\alpha+\beta)!(\alpha+m)!(\beta+m)!(a+b+2m)!a!b!}.$$

In all the applications of these theorems $a+b=\alpha+\beta$ so that

$$\begin{aligned} A_m &= 2^m A\frac{\alpha!(a+m)!(b+m)!}{(a+2m)!a!b!}, \\ B_m &= B\frac{\alpha!(a+2m)!}{(\alpha+2m)!a!}, \\ C_m &= C\frac{\alpha!\beta!(a+m)!(b+m)!}{(\alpha+m)!(\beta+m)!a!b!}. \end{aligned}$$

We may prove Theorem 3 by induction as follows. Assume that (158) holds for a certain value of m. Write

$$\begin{aligned}\alpha' &= \alpha + 2m, a' = a + m, b' = b + m, k' = k + m, n' = n + [k/4],\\ D_m &= \frac{(\alpha' + \beta + 2)(\alpha' + \beta + 1)}{(\alpha' + 2)(\alpha' + 1)}, E_m = \frac{(a' + b' + 1)(a' + b' + 2)}{2(a' + 1)(b' + 1)}.\end{aligned}$$

Then, by (33),

$$R_{\alpha'+2,\beta}(8n + 2k' + 2) = D_m \sum_{\mu=0}^{n'} R_{\alpha',\beta}(8n - 8\mu + 2k')R_{2,0}(8\mu + 2),$$

and

$$R_{a'+1,b'+1}(4n + k' + 1) = E_m \sum_{\mu=0}^{n'} R_{a',b'}(4n - 4\mu + k')R_{1,1}(4\mu + 1).$$

But

$$R_{2,0}(8\mu + 2) = R_{0,1}(4\mu + 1),$$

and it follows by induction, since $A_m D_m = A_{m+1} E_m$, and since (158) holds for $m = 0$.

Theorems 4 and 5 may be proved similarly.

Theorem 6 *As particular cases of Theorem* 3 *we have the following:*

(i) *If* $r > 0$,

$$R_{2r,0}(8n + 2r) = \frac{2^r (r!)^2}{(2r)!} R_{r,r}(4n + r). \tag{159}$$

(ii) *If* $r \geq 0$,

$$R_{2r,2}(8n + 2r) = \frac{2^{r-1} r!(r + 2)!}{(2r)!} R_{r,r+2}(4n + r).$$

Formula (159) was proved by Glaisher by elliptic function theory.

Theorem 7 *As particular cases of Theorem* 4 *we have the following:*

(i) *If* $r > 1$,

$$R_{2r-4,4} = \frac{(2r)!}{2.4!(2r - 4)!} R_{2r,0}(8n + 2r).$$

(ii) *If* $r \geq 0$,

$$R_{2r+4,4}(8n + 2r + 4) = \frac{8!(2r)!}{4!(2r + 4)!} R_{2r,8}(8n + 2r + 4).$$

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